Analytic solution to post-Newtonian accurate Kepler Equation

Maria Haney & Achamveedu Gopakumar

Theoretisch-Physikalisches Institut,
Friedrich–Schiller–Universität, Jena

- Project supported by

SFB/TR7, “Gravitationswellenastronomie”

funded by

the Deutsche Forschungsgemeinschaft (DFG)

The Newtonian 'Kepler problem': 1

- The Newtonian (N) two-body dynamics admits an analytic solution: the Keplerian parametrization (KP).

- In KP, the orbital motion of a binary occurs in a plane, with the relative separation vector $\mathbf{R} = (Rcos\phi, Rsin\phi, 0)$

- Using KP the radial motion is given by

$$R = a(1 - e\cos u)$$

The Newtonian 'Kepler problem': 2

- The parametrized angular motion reads

$$\phi - \phi_0 \quad = \quad v \equiv 2\,\mathrm{arctan}\left[\left(\frac{1+e}{1-e}\right)^{1/2} \tan\frac{u}{2}\right]$$

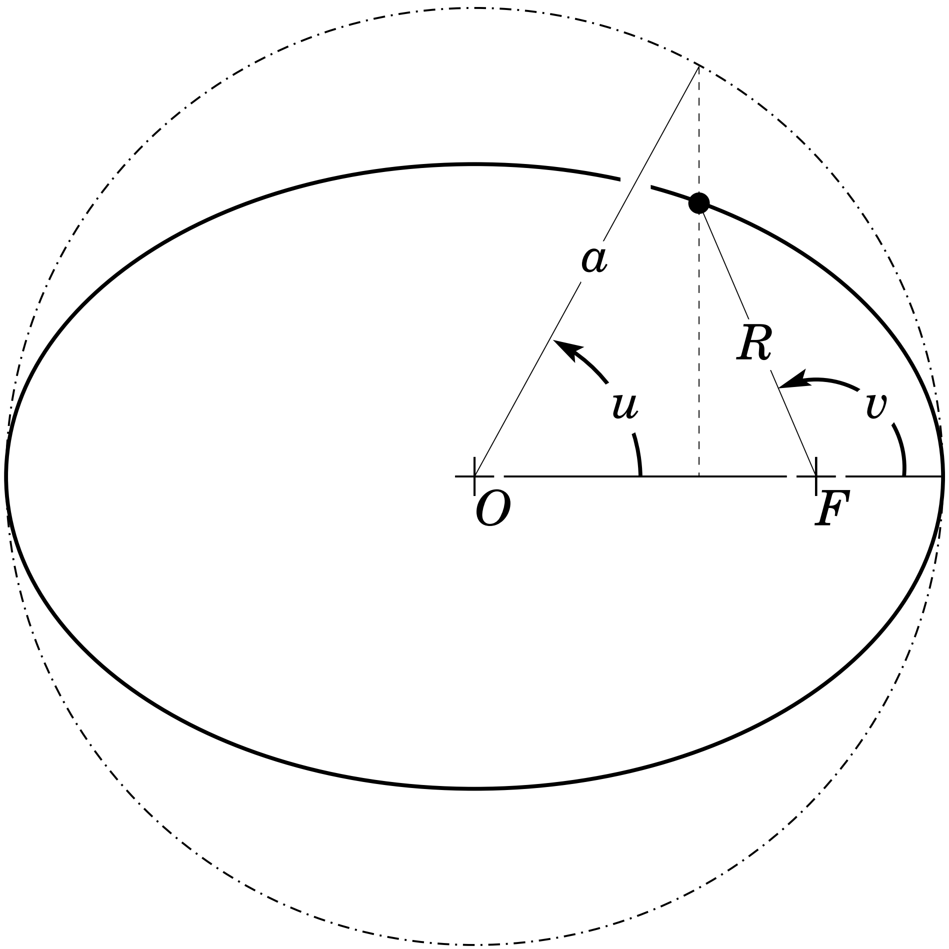
- The 'Kepler equation' (KE) connects the eccentric anomaly u to the coordinate time via

$$l \equiv n(t - t_0) \quad = \quad u - e\,\sin u$$

- The KP uses eccentric, true and mean anomaly u, v and l

- These anomalies have geometrical interpretations.

Orbital ellipse & auxiliary circle



Analytic solution to Newtonian KE

- For centuries the KE has been attracting the attention of mathematicians.
- Today there is a variety of both analytical and numerical solutions to Newtonian KE.
- A solution, credited to Bessel and in the form of a Fourier series of Bessel functions, reads

$$u = l + \sum_{k=1}^{\infty} \frac{2}{k} J_k(k\,e)\,\sin(k\,l)$$

- This talk deals with obtaining a similar solution to PN accurate Kepler equation.

PN accurate CB dynamics

- Post-Newtonian approximation provides corrections to Newtonian dynamics in terms of a small parameter $\nu \sim (v/c)^2 \sim (G\,m\,/c^2\,R)$
- The conservative motion of the compact binary is describable in terms of the PN accurate Hamiltonian

$$\mathcal{H}(\mathbf{r}, \hat{\mathbf{p}})|_{3\mathrm{PN}} = \mathcal{H}_0(\mathbf{r}, \hat{\mathbf{p}}) + \frac{1}{c^2}\mathcal{H}_1(\mathbf{r}, \hat{\mathbf{p}}) + \frac{1}{c^4}\mathcal{H}_2(\mathbf{r}, \hat{\mathbf{p}}) + \frac{1}{c^6}\mathcal{H}_3(\mathbf{r}, \hat{\mathbf{p}})$$

G. Schäfer and his collaborators (1984 - 2003)

- It is possible to derive a parametrization to 3PN accurate compact binary dynamics.

The 3PN accurate GQKP

- The 3PN accurate generalized quasi-Keplerian parametrization (GQKP) has been recently derived by Memmesheimer, Gopakumar, Schäfer (2004) and it extends earlier works of Damour, Deruelle, Schäfer & Wex.

- The radial motion of the compact binary is given by

$$r = a_r \left(1 - e_r \cos u\right)$$

- The angular motion reads

$$\varphi - \varphi_0 = (1+k)\,v + \left(\frac{f_{4\varphi}}{c^4} + \frac{f_{6\varphi}}{c^6}\right)\sin 2v + \left(\frac{g_{4\varphi}}{c^4} + \frac{g_{6\varphi}}{c^6}\right)\sin 3v + \frac{i_{6\varphi}}{c^6}\sin 4v + \frac{h_{6\varphi}}{c^6}\sin 5v$$

The 3PN accurate 'Kepler equation'

- In 3PN accurate GQKP the 'Kepler equation' reads

$$
\begin{aligned}
l \equiv n\,(t - t_0) \quad &= \quad u - e_t \sin u + \left(\frac{g_{4t}}{c^4} + \frac{g_{6t}}{c^6} \right) (v - u) \\
&\quad + \left(\frac{f_{4t}}{c^4} + \frac{f_{6t}}{c^6} \right) \sin v + \frac{i_{6t}}{c^6} \sin 2v + \frac{h_{6t}}{c^6} \sin 3v
\end{aligned}
$$

- The GQKP provides the PN orbital functions f_{4t}, f_{6t}, g_{4t}, g_{6t}, i_{6t}, h_{6t}, ... in terms of E, h and η and connects the eccentricities e_t, e_r and e_φ in a PN way.

- The PN accurate orbital elements and functions are coordinate dependent.

Analytic solution to 2PN accurate KE

- Analog to the Newtonian solution the 2PN accurate solution reads

$$u = \tilde{l} + \sum_{k=1}^{\infty} \frac{2}{k} J_k(k\,e)\,\sin(k\,\tilde{l})$$

- In harmonic coordinates $\tilde{l}$ in terms of l is given by

$$\tilde{l} = l - (\frac{GMn}{c^3})^{4/3} \left[\frac{15 - 6\eta}{2\sqrt{1 - {e_t}^2}} \left(v - u \right) + \frac{e_t}{8} \left(15 - \eta \right) \eta \frac{\sin u}{1 - e_t \cos u} \right]$$

- $(v - u)$ can be expressed as a continuous function of u

$$v - u = 2 \arctan \left[\frac{1 - \sqrt{1 - {e_t}^2}}{e_t - 1 - {e_t}^2 \cos u} \sin u \right]$$

- Plugging the Newtonian accurate solution $u(l)$ into $(v - u)$ and $\tilde{l}$ delivers a 2PN accurate u in terms of l.

- The 3PN accurate solution has been derived in a similar way.

Features

- Our solution depends on mean motion n, time eccentricity e_t and finite mass ratio η.
- The same solution is valid when spin effects are restricted to leading order spin effects.
- Our analytic solution formally shows that the above conservative PN accurate dynamics is fully integrable.

Future plans

- Determine numerically for which values of e_t the current solution holds good.

- Obtain PN accurate Hansen coefficients.